Remarks on non-standard statistics

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Abstract. The theory of non-standard statistics is reviewed and placed in a (co)homological setting. Some general mathematical results are shown to be relevant. The specific heat of a two-particle molecular gas is calculated as a function of temperature and the statistics determining parameter. It shows strong behaviour close to the fermion case. The second virial coefficient is calculated by a slightly improved method and its behaviour is compared with that of the vacuum energy of a quantum field interacting with a flux tube.

1. Introduction

The possibility of non-standard statistics in two dimensions arises, from a formal point of view, because the character group, π_1^*, of the fundamental group of the system configuration space, M, can differ from $\mathbb{Z}_2(\sim\Sigma_2$, the symmetric group of two elements).

This was effectively pointed out by Laidlaw and DeWitt in their path integral investigation [1] of quantum mechanics on topologically non-trivial spaces, and later by Leinaas and Myrheim [2] (see also Souriau [20]).

In dimensions d greater than two $\pi_1^* \sim \mathbb{Z}_2$ leading uniquely to Bose or Fermi statistics [1-3]. Whether one believes this to be the origin of the standard statistics or not, it is an attractive result and lends credence to the analyses of the two-dimensional case that have recently appeared. In the present work we should like to make some comments on these developments. First some standard formalism is required.

2. General formalism

One approach, but by no means the only one, to obtain the quantum mechanics on a multiply connected space M is by projection from the theory on the simply connected universal covering space $\tilde{M} = M/\Gamma$. Roughly speaking $\tilde{M}$ consists of $|\Gamma|$ copies of M related by the symmetry group Γ, which is isomorphic to the fundamental group of M.

In order that a wavefunction $\tilde{\psi}: \tilde{M} \to \mathbb{C}$ on $\tilde{M}$ should project down consistently to a wavefunction on M it is necessary that

$$\tilde{\psi}(\tilde{q}\gamma) = a(\gamma)\tilde{\psi}(\tilde{q}) \qquad \forall \gamma \in \Gamma,$$

where the $a(\gamma)$ form a one-dimensional unitary representation of Γ. Taking one of the copies in $\tilde{M}$ actually to be M, the wavefunctions on the other copies form the branches of a multivalued wavefunction on M, i.e. $\psi(q) = \tilde{\psi}(\pi^{-1}q)$ where $\pi: \tilde{q} \to q$ is the covering projection. Moving $\tilde{q}$ to $\tilde{q}\gamma$ on $\tilde{M}$ corresponds, when projected down, to q going round a loop on M. This will be a non-trivial loop if γ is not the identity.

All this is very basic. Some of the original references are given in [4] (see also [5]).

Assume now that, for convenience, ψ obeys the free Schrödinger equation on $\tilde{M}$. Then so will ψ on M through the (position independent) projection π. However $\psi(q)$ is multivalued. It is possible, quite generally, to remove this multivaluedness at the expense of introducing an interaction. The way this comes about is as follows [4].

The branches of the wavefunction are in one-to-one correspondence with the elements of π_1, i.e. with the homotopy classes of loops on M. However since we are dealing with scalar, U(1) quantum mechanics, or functional quantum field theory, the representation $a(\gamma)$ is Abelian and the homotopy classes of loops can be replaced by homology classes of cycles. The first homology group $H_1(M, \mathbb{Z})$ is the Abelianised π_1. Those branches which correspond to different γ's mapping to the same homology group element, h, will have the same phase factor, $a(h)$ say. That is,

$$\tilde{\psi}(\tilde{q}\gamma) = a(h)\tilde{\psi}(\tilde{q}) \tag{1}$$

$\forall \gamma$ such that $\gamma \to h$ under $\pi_1 \to H_1$.

The $a(h)$ form a unitary representation of H_1. This has two advantages. Firstly, in any specific case, we can bypass the fundamental group and move directly to H_1 which, as a rule, is easier to obtain. Secondly, in the general case, we can employ the structure theorem

$$H_1(M, \mathbb{Z}) \sim b_1\mathbb{Z} \oplus \sum_{i=1}^{k_1} \mathbb{Z}_{t_i^1}$$

i.e. a sum of $b_1\mathbb{Z}$'s, the free part, and k_1 finite cyclic groups, the torsion part (b_1 is the first Betti number and the t_i^1 the first torsion numbers).

Consider the case with no torsion. The one-dimensional representations $a(h)$ are

$$a(h) = a(n) = \exp(2\pi i \boldsymbol{\alpha} \cdot \boldsymbol{n})$$

where the b_1 dimensional vector $\boldsymbol{n} = \{n^m\}$ ($n^m \in \mathbb{Z}$) labels the homology class and the real vector $\boldsymbol{\alpha} = \{\alpha_m\}$ ($\alpha_m \in \mathbb{R}$) labels the representation. Since $a(h)$ is a phase we aim to remove it by a gauge transformation. De Rham's first theorem ensures the existence of a real, closed and regular 1-form $\omega(\boldsymbol{\alpha})$ on M (assumed orientable) with periods $\{\alpha_m\}$

$$\oint_{e_m} \omega(\boldsymbol{\alpha}) = \alpha_m$$

over the fundamental cycles labelled by the unit vectors $e_m = (0, 0, \ldots, 1, \ldots, 0)$ with the 1 in the mth position. The general cycle is labelled by $\boldsymbol{n} = n^m e_m$ so that

$$a(\boldsymbol{n}) = \exp\left(2\pi i \oint_{\boldsymbol{n}} \omega(\boldsymbol{\alpha})\right). \tag{2}$$

Choose now some fiducial point q_0 and a reference path from q_0 to the current point q. Then we can label all paths from q_0 to q by the homology class of the cycle formed from a path together with the reference path from q to q_0, and hence by the vector $\boldsymbol{n}$. We choose the reference path to correspond to the homology class $\boldsymbol{n} = \boldsymbol{0}$. With obvious notation the loop integral is split into a part over the reference path and a part over the remaining path, which is labelled now by $\boldsymbol{n}$

$$\oint_{\boldsymbol{n}} \omega(\boldsymbol{\alpha}) = \int_{q_0}^q \omega(\boldsymbol{\alpha})\bigg|_{\boldsymbol{n}} - \int_{q_0}^q \omega(\boldsymbol{\alpha})\bigg|_0. \tag{3}$$

Therefore, if we effect the phase transformation

$$\psi'(q) = \exp\left(-2\pi i \left.\int_{q_0}^{q} \omega(\boldsymbol{\alpha})\right|_{\mathbf{0}}\right)\psi(q),$$ (4)

it is clear that as q goes round a loop (or rather a cycle) the phase factor will cancel any multivaluedness making $\psi'(q)$ single valued. Spelling things out, if $\tilde{q} \to \tilde{q}\gamma$ is the lift of the loop and if γ maps to $\boldsymbol{n}$ under $\pi_1 \to H_1$ then $\mathbf{0}$ in the phase factor in (4) is replaced by $\boldsymbol{n}$ and $\psi(q)$ by $a(\boldsymbol{n})\psi(q)$, according to (1). Equations (2) and (3) guarantee that the combination on the right-hand side of (4) is unchanged.

The gauge transformation modifies the derivative with ω acting as a vector potential form. If $\nabla \equiv dq^i \nabla_i$ we have

$$(\nabla + 2\pi i \omega)\psi' = \exp\left(-2\pi i \left.\int_{q_0}^{q} \omega\right|_{\mathbf{0}}\right)\nabla\psi.$$

In Lagrangian terms it corresponds to replacing L by $L - 2\pi\omega(\boldsymbol{\alpha})/dt$. Because ω is closed, $d\omega = 0$, the classical equations of motion are unaffected.

As a physical realisation one might imagine $\omega(\boldsymbol{\alpha})$ to be due to a series of b_1 magnetic fields, with fluxes proportional to the periods α_m, passing through the fundamental cycles e_m, as in the Aharonov–Bohm effect [4, 6–7].

It is thus seen that the multivaluedness has been eliminated, at the cost of introducing an interaction. Of course there is no need to transfer all the multivaluedness into an interaction (or vice versa). A mixed situation is quite possible and, in the Aharonov–Bohm case, was discussed a long time ago by Kretzschmar [7].

The general idea is very old and probably dates back to Weyl [8]. Dirac [9] gives the standard discussion of the electromagnetic case and Eddington makes statements of a similar nature in [10]. Pandres [11] outlines a more general scheme. In the particular context of the Aharonov–Bohm effect, Kretzschmar [7] gives a very instructive treatment and Byers and Yang [12] provide an exemplary statement of the facts in connection with flux quantisation. The replacement of a topological restriction by an interaction appears explicitly in the work of Edwards [13] on polymer statistical mechanics.

3. Statistics

So far this is a very general scheme. In order to apply it to statistics one has only to decide on the configuration space and then to investigate its homology. First some well known, scene-setting facts.

For a system of N particles moving on a d-dimensional space, M_d, the conventional configuration space is the N-fold product $(M_d)^N$. If the particles are identical, the statistics, Bose or Fermi, are normally imposed through the symmetry nature of the wavefunctions.

The idea here is different. The configuration space itself is made to reflect the identity by taking the factor of $(M_d)^N$ by the permutation group on N symbols Σ_N, i.e. by identifying points on $(M_d)^N$ that differ only by having two, or more, particle coordinates interchanged. If $d > 2$ then $(M_d)^N/\Sigma_N$ is not a manifold and one has to remove the diagonals (the place, Δ, where two or more coordinates are the same) to make it into one. The configuration space M is then $C_N(M_d) \equiv (M_d^N - \Delta)/\Sigma_N$, a well

known mathematical construction. If d is greater than two, taking out the diagonals does no damage to the loop topology of M_d^N so that $M_d^N - \Delta$ is simply connected (if M_d is). Hence $\pi_1(M) = \Sigma_N$ for $d > 2$. The Abelianised Σ_N is $\mathbb{Z}_2$ with the even permutations mapping to the unit element of $\mathbb{Z}_2$ and the odd ones to its generator. The two representations of $\mathbb{Z}_2$ then yield Bose and Fermi statistics [1].

If d equals two there is less room for manoeuvre and $M_2^N - \Delta$ is not simply connected. For two particles in the plane $M_2^N \sim R^4$ and Δ is a plane, R^2. $R^4 - \{\text{plane}\}$ has the same loop structure as its successive deformation retracts, $R^3 - \{\text{line}\}$, $R^2 - \{\text{point}\}$ and S^1. Therefore the fundamental group for two distinguishable particles in the plane is $\mathbb{Z}$. Dividing by Σ_2, to make the particles identical, does not change the fundamental group which is still $\mathbb{Z}$ but, in this case, it is the group of all the integers (corresponding to a π rotation) while for distinguishable particles it is the group of even integers (corresponding to a 2π rotation). Removal of the inessential centre-of-mass degree of freedom can be taken as the deformation retract leading to the space $R^2 - \{\text{point}\}$ which is topologically a cone. Dividing by Σ_2 makes this a cone of opened angle π, [2].

In the general case, for N identical particles, the fundamental group is the braid group, [14]. For distinguishable particles it is the coloured braid group. Until § 5 we shall take M_d to be the plane R^2, which is the most often studied case.

According to our general result of the previous section we do not need the fundamental group. The first homology group would do. On the plane this will be generated by circles, of which there are $\frac{1}{2}N(N-1)$ since each particle can circle around every other. Therefore $H_1(R^{2N} - \Delta; \mathbb{Z}) \sim \frac{1}{2}N(N-1)\mathbb{Z}$ and there is no torsion. If the particles are identical every $\mathbb{Z}$ becomes equivalent and $H_1(C_N(R^2); \mathbb{Z}) \sim \mathbb{Z}$, a result that also easily follows by Abelianising the braid group.

It is thus seen that the quantum mechanics of N distinguishable particles on the plane is labelled by $\frac{1}{2}N(N-1)$ real parameters $\boldsymbol{\alpha}$. Dividing by Σ_N forces these parameters to be the same and so, for identical particles, one needs just one parameter, exactly like the elementary theory of quantum mechanics on the circle [15] or the Aharonov-Bohm effect [6]. In these cases the form ω is the angle form $\alpha\, d\varphi/2\pi$ and it is clear that in the present case there will be a similar term in ω for every pair of particles, the angle being the relative angle of rotation of the pair.

The braid group appears in the theory of Riemann surfaces [17] and it is convenient to use complex coordinates $z_i = x_i + iy_i$ $(i = 1, \ldots, N)$ for the particle coordinates. Then one can write

$$\omega(\boldsymbol{\alpha}) = \frac{1}{\pi} \sum_{i<j} \alpha_{ij}\, d\varphi_{ij} \tag{5}$$

in the distinguishable case, and

$$\omega(\boldsymbol{\alpha}) = \frac{\alpha}{\pi} \sum_{i<j} d\varphi_{ij}$$

for identical particles, where $z_i - z_j = r_{ij} \exp(i\varphi_{ij})$.

Although this result is obvious it is possible to obtain it directly from known facts about the cohomology of configuration spaces, which is what is being used through De Rham's theorem.

The cohomology in the distinguishable case has been fully worked out by Arnold [18] and Brieskorn [19] (see also [16]). The results are that (i) there is no torsion in

any cohomology dimension, (ii) the Poincaré polynomial is $(1 + t)(1 + 2t) \ldots (1 + (N-1)t)$, from which the Betti numbers can be read off, e.g. $b_1 = \frac{1}{2}N(N-1)$, $b_2 = \frac{1}{24}N(N-1)(N-2)(3N-1)$, (iii) the cohomology vanishes in dimensions greater than the complex dimension of the space ($=$ number of particles) and (iv) the cohomology ring is isomorphic to the algebra generated by the regular 1-forms

$$\omega_{ij} = \frac{1}{2\pi i} \frac{d(z_i - z_i)}{z_i - z_j} \qquad (i < j).$$

These forms have the same function as the angle forms $d\phi_{ij}/2\pi$ and so we regain the previous result in a mathematically rigorous fashion. In place of (5) one now has

$$\omega(\boldsymbol{\alpha}) = 2 \sum_{i<j} \alpha_{ij} \omega_{ij}. \tag{6}$$

The cohomology in the identical case is harder to evaluate. However the free part of H^* has been determined to be trivial in all but the first two dimensions, the first Betti number being unity, i.e. $H^1(C_N(R^2); \mathbb{Z}) \sim \mathbb{Z}$. This of course agrees with the fact that H^1 is isomorphic to the free part of H_1. The form $\omega(\boldsymbol{\alpha})$ is obtained from (6) by setting all the α_{ij} equal,

$$\omega(\boldsymbol{\alpha}) = 2\alpha \sum_{i<j} \omega_{ij}$$

as before.

This is all that is needed here but a few more general facts may be interesting.

Because $H_1 \sim \mathbb{Z}$, H^2 is torsion free and hence trivial. (This is not true in the coloured case.) Thus, according to the classification theorem of Cartan, Kostant, Souriau and Isham [20], there are no non-trivial $U(1)$ bundles over the configuration space of identical particles in the plane. This means that the classification of the quantum mechanics by the vector $\boldsymbol{\alpha}$ is complete. If the dimension of M_d is greater than two and $\pi_1(M_d)$ is trivial we have seen that $H_1 \sim \mathbb{Z}_2$. Further it can be shown that the free part of H_2 is trivial; hence $H^2 \sim \mathbb{Z}_2$ and the two $U(1)$ bundles correspond to Bose and Fermi statistics [21]. Quite generally, since the torsion parts of H_1 and H^2 are the same, the different $U(1)$ bundles corresponding to $\mathrm{Tor}(H^2)$ are accounted for in the description of the possible multivalued wavefunctions according to the representations of H_1 given earlier. Any incompleteness in this latter will be due to the existence of a free part to H^2.

For large numbers of particles the cohomology of $C_N(R^2)$ stabilises and in dimensions above two appears to be all torsion (see, e.g., [19]).

The use of (6) means that the phase transformation (4) to a single valued wavefunction ψ' reads, in complex notation,

$$\psi'(z_i, z_i^*) = \prod_{i<j} (z_i - z_j)^{-2\alpha_{ij}} \psi(z_i, z_i^*)$$

in the distinguishable case, while for identical particles one gets

$$\psi'(z_i, z_i^*) = \left(\prod_{i<j} (z_i - z_j) \right)^{-2\alpha} \psi(z_i, z_i^*).$$

The constant phase factor corresponding to the fiducial point q_0 has been dropped.

These formulae appear in some recent papers [22] but we see them here as examples of a general formalism [4]. They can be given a more symmetrical form by writing

(we give only the identical case)

$$\omega(\alpha) = \alpha \sum_{i<j} (\omega_{i_j} + \omega_{i_j}^*)$$

so that now

$$\psi'(z_i, z_i^*) = \prod_{i<j} \left(\frac{z_i^* - z_j^*}{z_i - z_j} \right)^\alpha \psi(z_i, z_i^*)$$

$$\equiv \Delta^{-\alpha} \psi(z_i, z_i^*).$$

The derivatives are modified according to

$$(\mathrm{d}/\mathrm{d}z_i - \alpha A_i)\psi' = \Delta^{-\alpha}(\mathrm{d}/\mathrm{d}z_i)\psi$$

$$(\mathrm{d}/\mathrm{d}z_i^* + \alpha A_i^*)\psi' = \Delta^{-\alpha}(\mathrm{d}/\mathrm{d}z_i^*)\psi \tag{7}$$

where

$$A_i = \sum_{k \neq i} (z_i - z_k)^{-1}.$$

Expressed in complex variables the construction of the configuration space is as follows.

The action of the transpositions of the permutation group Σ_N on the coordinates z_i is to reflect the point of $\mathbb{C}^N$ in one of the hyperplanes $z_i = z_j$. The collection of these $\frac{1}{2}N(N-1)$ hyperplanes is characterised by the equation $\prod_{i<j}(z_i - z_j) = 0$ and makes up the $(2N-2)$-dimensional space Δ which forms the 'branching space' of the branched covering $\mathbb{C}^N \rightarrow \mathbb{C}^N/\Sigma_N$ ([17] II § 2). The excision of Δ from $\mathbb{C}^N$ yields the unbranched covering $(\mathbb{C}^N - \Delta) \rightarrow (\mathbb{C}^N - \Delta)/\Sigma_N$.

It is interesting to note that the whole formalism can be generalised (e.g. [19]). For example, Σ_N can be replaced by the Weyl group of a semisimple Lie group and the cohomology generating 1-forms are given in terms of the 1-forms describing the planes of reflection. It is not known whether these generalisations have any physical relevance.

The extreme cases of $\alpha = 0$ and $\alpha = \frac{1}{2}$ correspond to Bose and Fermi statistics. A physical realisation of the phase change can be given in which each particle possesses both charge and a magnetic flux [23]. These particles can reasonably be called dyons because a magnetic flux tube in three dimensions is the analogue of a magnetic monopole in two (cf [24] § 2).

4. Statistical mechanics

Except in the cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ or $N = 2$ it is, apparently, difficult to implement the multivaluedness or, equivalently, to solve Schrödinger's equation for a single valued wavefunction with the vector potential interaction (7).

The two-particle system can be analysed. One possibility is to determine the dependence on α of the specific heat of a dilute gas of 'diatomic' molecules. An unspecified two-body binding force is assumed to fix the interparticle distance, and the mass of the particles is taken to be such that the internal partition function is

$$Z = \sum_{m=-\infty}^{\infty} \exp[-(m+\alpha)^2/T] = (\pi T)^{1/2}\theta_3(\alpha, i\pi T),$$

a theta function ([35], p 138).

The results for the specific heat C_V are shown in figure 1. For large temperatures, T, C_V approaches the classical value of $\frac{1}{2}$, from below if $\frac{1}{4} < \alpha < \frac{1}{2}$ and from above if $0 < \alpha < \frac{1}{4}$. The $\alpha = 0$ and $\alpha = \frac{1}{2}$ curves are similar to those for ortho- and para-hydrogen, as might have been expected. However for α between $\frac{1}{4}$ and $\frac{1}{2}$ there is a maximum at smallish T. If we set $\bar{\alpha} = \frac{1}{2} - \alpha$ with $\bar{\alpha}$ small then, for smallish T, C_V is well approximated by

$$C_V \sim (\bar{\alpha}/T)^2 (\cosh(\bar{\alpha}/T))^{-2}.$$

No matter how close α is to $\frac{1}{2}$, C_V always has a maximum of ~ 0.439 at $T \sim 0.913\,\bar{\alpha}$. C_V is periodic in α with period one and is symmetric about $\alpha = \frac{1}{2}$. It is plotted in figure 2 for the value $T = 0.05$. As T decreases the peak becomes sharper.

Another peculiar behaviour in interpolating statistics has been discussed by Arovas *et al* [25]. They calculate the second virial coefficient for a dilute gas of particles obeying non-standard statistics. This is possible because the coefficient depends on only the two-particle partition function. Arovas *et al* calculate this latter quantity in two ways, the second of which they call a path integral method. Leaving this point aside and the fact that similar expressions are to be found in the work of Edwards [13] and Kretzschmar [7], we give here another, related treatment which has certain technical advantages.

The standard formula for the second virial coefficient $B(T)$ for two-dimensional problems is

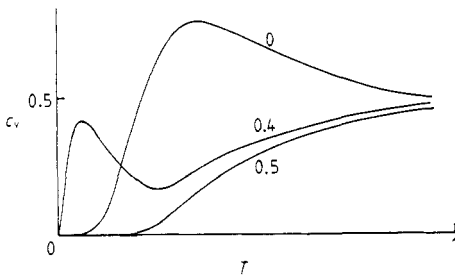$$B(T) = \lim_{A \to \infty} (\tfrac{1}{2} A - 2\lambda^2 Z(2)) \tag{8}$$

Figure 1. The specific heat C_V as a function of temperature T for three values, 0, 0.4 and 0.5, of the statistics determining parameter α. Units are $\hbar = c = k_B = 1$.
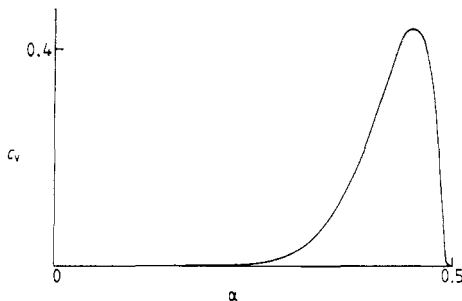

Figure 2. The specific heat C_V as a function of the statistics determining parameter α for fixed $T = 0.05$.

where A is the surface area, $Z(2)$ the relative motion partition function and $\lambda^2 = 2\pi/mT$. The leading term in $Z(2)$ is formally divergent and is expected to cancel the area term in (8). We want the finite remainder.

To evaluate $Z(2)$ we use the expression, [26], for the quantum mechanical propagator $K_{\alpha,\beta}(r, r', \tau)$ on a cone of opened angle β and with phase factor $\exp(2\pi i\alpha)$. If β is set equal to π then we obtain the interchange of two particles by going around the cone once. (This cone is the configuration space of two identical particles after the removal of the topologically, *and* dynamically, trivial centre-of-mass dimension [2].)

In terms of this propagator the relative partition function is given by

$$Z(2) = \int K_{\pi,\alpha}(r, r, -i/mT) \, dr \tag{9}$$

where the polar angle of r runs from 0 to π.

Rather than the Bessel function series for $K_{\beta,\alpha}$ it is better to use the contour integral expression given in [26]

$$K_{\pi,\alpha}(r, r', \tau) = -\frac{1}{8\pi^2\tau} \int_\Gamma \exp[i(\boldsymbol{\rho}-\boldsymbol{\rho}')^2/4\tau]$$
$$\times \frac{\exp[i(\zeta - \varphi + \varphi')(2\alpha-1)]}{\sin(\zeta - \varphi + \varphi')} \, d\zeta, \qquad 0 < \alpha \le 1,$$

where $r = (r, \varphi)$, $r' = (r', \varphi')$, $\boldsymbol{\rho} = (r, \zeta)$ and $\boldsymbol{\rho}' = (r', 0)$. The contour Γ is shown in [26]. The most convenient choice for our purposes is a closed contour around the pole at $\zeta = (\varphi - \varphi')$ together with the infinite, oppositely directed vertical lines $\zeta = \varphi - \varphi' \pm \pi/2 + iy$, $-\infty < y < \infty$. The pole gives a contribution equal to the usual propagator on the plane

$$-\frac{i}{4\pi\tau} \exp\left(\frac{i(r'-r)^2}{4\tau}\right),$$

whatever the value of β. It is easily checked that, using (9), this cancels the area term in (8), as expected, exactly as in the field theory calculation of the one-loop effective Lagrangian. The contribution of the vertical lines is the sought-for finite correction. A very easy calculation gives the result

$$B(T) = -\lambda^2[2(\tfrac{1}{2} - \alpha)^2 - \tfrac{1}{4}]$$

agreeing with Arovas *et al.* This formula has to be extended by periodicity to $\alpha > 1$ and then shows cusps when α is an integer.

A similar behaviour is found in the vacuum energy (density) of a quantum field in the presence of a flux tube, [27]. On the spacetime $T \times S^1$ a complex field with circulating phase change $\exp(2\pi i\alpha)$ has a vacuum energy $E = \tfrac{1}{12} - (\tfrac{1}{2} - \alpha)^2$, if $0 \le \alpha \le 1$. This formula too must be extended by periodicity and shows the same behaviour as $B(T)$. Similar calculations can be performed in higher dimensional spatial geometries resulting in higher polynomials in $(\tfrac{1}{2} - \alpha)$. Thus, in the spacetime $T \times (R^2 - \{\text{point}\})$, the one-loop effective Lagrangian density is $\bar{\alpha}(1 - \bar{\alpha}^2)/48\pi r^3 \tan(\pi\bar{\alpha})$ while in $T \times (R^3 - \{\text{line}\})$ it is $\alpha(1 - \alpha^2)(2 - \alpha)/24\pi^2 r^4$ where r is the distance from the singularity.

5. Further formalities

It has been suggested that the quasiparticle excitations that occur in some explanations of the quantum Hall effect obey fractional statistics. In one discussion, [28], of this

effect the single-particle configuration space is taken to be the two-sphere, the necessary magnetic field being produced by a central monopole. It is therefore relevant to consider the configuration spaces $[(S^2)^N - \Delta]/\Sigma_N$. Fortunately the corresponding braid groups have been analysed [29]. Abelianisation, which amounts to assuming that everything commutes, easily shows that the first homology group has one generator, σ, subject to the relation $\sigma^{2N-2} = 1$, i.e. $H_1 \sim \mathbb{Z}_{2N-2}$. This means that the statistics can depend on the number of particles present. For two particles only bosons and fermions are possible. For any N the representations $a(\gamma)$ of $\mathbb{Z}_{2N-2}$ are generated by the $(2N-2)$th roots of unity, $1, \omega, \omega^2, \ldots, \omega^{2N-3}$. The boson representation is generated by 1 and the fermion one by ω^{N-1}. The homology classes are labelled by an integer $n = 0, 1, \ldots, (2N-3)$ and the phase factors are $a(n) = \exp[2\pi i r n/(2N-2)]$ in the representation generated by ω^r. We automatically obtain fractional statistics, but with even denominators. A similar conclusion is reached in [30].

As before, this conclusion can be reached without mentioning the braid group by looking directly at the cohomology of the configuration space. We do not go into details but refer to [31] and [32].

The only way to avoid number dependent statistics is to choose the boson or fermion case.

6. Comments and conclusion

The intention of the present work is, in part, to place the recent discussions of non-standard statistics in a more general setting and to draw attention to the relevance of homology theory.

One outstanding problem is to obtain a quantum field theory incorporating non-standard statistics. Even for three particles there are difficulties not present for two, [22-23].

Incidentally it is amusing to note that by deformation retractions the configuration space of three particles can be reduced to a homological dimension of two, namely to the space of isosceles triangles in the plane with a fixed scale and a fixed centroid (see the interesting paper of Bloore *et al* [33], and [31], for the corresponding three-space case). This space is not a manifold. In the distinguishable case it looks roughly like two tori with three circumferences in common (and further identifications). For N particles the corresponding series of retractions shows that the top, non-trivial (co)homology dimension is $N - 1$ and not N.

Unfortunately we have nothing to say about the more than two-particle problem. The three-particle propagator can be reperiodised in the relative angles but the contour integral does not appear tractable.

Concerning the vacuum energy of a quantum field in the presence of a flux tube, mentioned at the end of § 4, Ford [34] has evaluated the energy density for twisted scalar fields in three-dimensional space. This corresponds to the $\alpha = \frac{1}{2}$ case here.

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